Separation of variables on a non-hyperelliptic curve

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1 Introduction

In the paper we consider several models that admit a separation of variables on the following algebraic curve of genus 4:

$$\Phi(\xi, Y) = s_6 Y^6 + l(\xi) Y^4 + k(\xi) Y^2 - S(\xi) = 0, \tag{1.1}$$

where

$$s_6 = \frac{\delta^6}{6!} S^{VI}(\xi), \qquad k(\xi) = \frac{\delta^2}{10} S''(\xi) + 4(\alpha\xi^2 + e_2\xi + e_1), \qquad l(\xi) = \frac{\delta^4}{4!} S^{IV}(\xi) - \frac{\delta^2}{2} k''(\xi). \tag{1.2}$$

Here α, δ are parameters, S is arbitrary sixth degree polynomial. If $\delta = 0$, then the curve is hyperelliptic of genus 3.

The class of curves (1.1) with $\delta \neq 0$ can be described as follows. Let $P(\eta, \xi) = 0$ be an arbitrary cubic. In the generic case it is an elliptic curve. Then (1.1) is a double cover over the cubic defined by the formula $\eta = \xi^2 - \delta^2 Y^2$.

To present out approach, we start with the Hamiltonian

$$H = a p_1^2 + c p_2^2 + d p_1 + e p_2 + f, \tag{1.3}$$

where

$$a = -\frac{4 s_2 S(s_1)}{s_1 - s_2}, \qquad c = \frac{4 s_1 S(s_2)}{s_1 - s_2}, \qquad d = -s_1 \frac{J}{s_1 - s_2}, \qquad e = -s_2 \frac{J}{s_1 - s_2},$$

$$\begin{gathered} f = \frac{\delta^2}{40} \frac{s_2 S''(s_1) - s_1 S''(s_2)}{s_1 - s_2} - \frac{\delta^2}{4} \frac{s_2 S'(s_1) + s_1 S'(s_2)}{(s_1 - s_2)^2} + \\ \frac{3\delta^2}{4} \frac{s_2 S(s_1) - s_1 S(s_2)}{(s_1 - s_2)^3} + \alpha s_1 s_2, \end{gathered} \tag{1.4}$$

and

$$J = 2\delta \frac{\sqrt{S(s_1)}\sqrt{S(s_2)}}{s_1 - s_2}$$

as the most general model. It can be easily verified that H commutes with the function

$$K = Ap_1^2 + Cp_2^2 + Dp_1 + Ep_2 + F, \tag{1.5}$$

where

$$A = \frac{4S(s_1)}{s_1 - s_2}, \qquad C = -\frac{4S(s_2)}{s_1 - s_2}, \qquad D = \frac{J}{s_1 - s_2}, \qquad E = \frac{J}{s_1 - s_2},$$

$$F = \frac{\delta^2}{40}\frac{S''(s_2) - S''(s_1)}{s_1 - s_2} + \frac{\delta^2}{4}\frac{S'(s_1) + S'(s_2)}{(s_1 - s_2)^2} + \frac{3\delta^2}{4}\frac{S(s_2) - S(s_1)}{(s_1 - s_2)^3} - \alpha(s_1 + s_2) \tag{1.6}$$

with respect to the standard Poisson bracket $\{p_\alpha, s_\beta\} = \delta_{\alpha\beta}$. If $\delta = 0$, then the terms in H and K linear in momenta disappear and H belongs to the Stäckel class. In this case s_1, s_2 are separated variables. The corresponding Abel's transformation on the hyperelliptic curve of genus 3 has the form

$$\frac{ds_1}{\sqrt{T(s_1)}} + \frac{ds_2}{\sqrt{T(s_2)}} = dt, \qquad \frac{s_1 ds_1}{\sqrt{T(s_1)}} + \frac{s_2 ds_2}{\sqrt{T(s_2)}} = 0.$$

Here

$$T(s) = 16\, S(s)\, (\alpha s^2 + e_2 s + e_1),$$

and the constants e_1 and e_2 are values of the integrals H and K, respectively.

The main result of the paper is an explicit formula for the action in the case $\delta \neq 0$ depending on two parameters. We find it directly from the Hamilton-Jacobi equation.

In Sections 4-6 we find the action and a separation of variables for the Kowalewski hyrostat, the Clebsch and the $so(4)$ Schottky-Manakov spinning tops in a way analogous to that used in Sections 2-3 for the Hamiltonian (1.3). Many significant papers (see, for instance, [1, 4, 6, 2, 3, 18, 19, 7, 11, 12, 13, 5, 21] and references there-in) are devoted to these models. In particular, it is known [20, 21]) that the models are linked via some changes of variables. However according to a common opinion of experts, a "satisfactory" separation of variables for the above models is still not found.

For the Clebsch and the $so(4)$ Schottky-Manakov tops S is a polynomial of degree 4. An appearance of the corresponding algebraic curve of genus 3 could be predicted: the characteristic curve for the 3×3-Lax operator found by A. Perelomov [5] for the Clebsch top has the same form. However in this paper we don't use any Lax representations and the curve arises during computations in a rather natural manner.

The case $S =$ const also is of interest. The inhomogeneous hydrodynamic type system (2.7) corresponding to the pair (1.3), (1.5) coincides with the Gibbons-Tsarev equation [9].

The separation of variables found in this paper gives rise to a family of elliptic solutions for the Gibbons-Tsarev equation (see Section 7).

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2 Equations of motion

The Hamilton equations corresponding to (1.3) and (1.5) have the form

$$\frac{ds_1}{dt} = 2ap_1 + d, \qquad \frac{ds_2}{dt} = 2cp_2 + e. \tag{2.1}$$

and

$$\frac{ds_1}{d\tau} = 2Ap_1 + D, \qquad \frac{ds_2}{d\tau} = 2Cp_2 + E. \tag{2.2}$$

Finding p_1, p_2 from (2.1) and substituting them in the relations $H = e_1$ and $K = e_2$, we get

$$(s_1 - s_2)[s_1 \frac{\dot{s_1}^2}{S(s_1)} - s_2 \frac{\dot{s_2}^2}{S(s_2)}] + 16(e_1 - f)s_1 s_2 + \frac{4\delta^2}{(s_1 - s_2)^3}(s_2^3\, S(s_1) - s_1^3\, S(s_2)) = 0 \tag{2.3}$$

and

$$(s_1 - s_2)s_1 s_2 \left(\frac{\dot{s_1}^2}{S(s_1)} - \frac{\dot{s_2}^2}{S(s_2)} \right) - 4\delta \sqrt{S(s_1)S(s_2)} \left(s_1^2 \frac{\dot{s_1}}{S(s_1)} + s_2^2 \frac{\dot{s_2}}{S(s_2)} \right) + \tag{2.4}$$

$$16 s_1 s_2((e_1 - f)(s_1 + s_2) + (e_2 - F)s_1 s_2) - \frac{4\delta^2}{(s_1 - s_2)^3} \left(s_2^3\, S(s_1)(s_1 - 2s_2) - s_1^3\, S(s_2)(s_2 - 2s_1) \right) = 0,$$

where f and F are defined by (1.4), (1.6).

One of the basic technical problems is a rewriting of this system in a compact form. The straightforward elimination say $\dot{s_2}$ leads to a cumbersome equation of degree 4 with respect to $\dot{s_1}$.

Proposition 1. *Let (u, v) be any solution of the following system of equations:*

$$L(s_2)u^2 - 2L(s_1)v - M(s_2, s_1) = 0, \qquad L(s_1)v^2 - 2L(s_2)u - M(s_1, s_2) = 0, \tag{2.5}$$

where

$$M(x, y) = 3L(y) + L'(y)(x - y) + k(y)(x - y)^2, \qquad L(x) = \delta^2 S(x).$$

Then the derivatives

$$\dot{s}_1 = -J \frac{s_2\, u + s_1}{s_1 - s_2}, \qquad \dot{s}_2 = -J \frac{s_1\, v + s_2}{s_1 - s_2} \tag{2.6}$$

satisfy the system (2.3), (2.4).

The elimination of variables p_1, p_2 from (2.1), (2.2) leads to the inhomogeneous hydrodynamic type system

$$(s_1)_t + s_2(s_1)_\tau = -J, \qquad (s_2)_t + s_1(s_2)_\tau = J. \tag{2.7}$$

It follows from this formula and from (2.6) that

$$(s_1)_\tau = J\frac{u+1}{s_1 - s_2}, \qquad (s_2)_\tau = J\frac{v+1}{s_1 - s_2}. \tag{2.8}$$

Moreover, we find from (2.6) that

$$p_1 = \frac{J\,u}{8S(s_1)}, \qquad p_2 = -\frac{J\,v}{8S(s_2)}. \tag{2.9}$$

The following parameterization for solutions of (2.5) turns out to be crucial for the separation of variables.

Proposition 2. *Let z_1, z_2, z_3 be solutions of the equation*

$$z^6 L(s_2)^2 - L(s_2)M(s_2, s_1)z^4 + M(s_1, s_2)L(s_1)z^2 - L(s_1)^2 = 0 \tag{2.10}$$

such that

$$\frac{L(x)}{L(y)} = z_1 z_2 z_3, \qquad \frac{M(y,x)}{L(y)} = z_1^2 + z_2^2 + z_3^2, \qquad \frac{M(x,y)}{L(x)} = \frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2}.$$

Then

$$\begin{aligned}(u_1, v_1) &= (z_1 + z_2 + z_3,\ \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}), \qquad (u_2, v_2) = (z_1 - z_2 - z_3,\ \frac{1}{z_1} - \frac{1}{z_2} - \frac{1}{z_3}),\\ (u_3, v_3) &= (z_2 - z_1 - z_3,\ \frac{1}{z_2} - \frac{1}{z_1} - \frac{1}{z_3}), \qquad (u_4, v_4) = (z_3 - z_1 - z_2,\ \frac{1}{z_3} - \frac{1}{z_1} - \frac{1}{z_2})\end{aligned} \tag{2.11}$$

are solutions of (2.5).

3 Separation of variables.

To separate the variables in the case $\delta \neq 0$, we find the action function $\tilde{S}(s_1, s_2)$ in an explicit form.

Let us consider the system of equations

$$\Phi(\xi, Y) = 0, \qquad Y^2 = \frac{1}{\delta^2}(\xi - s_1)(\xi - s_2),$$

where the polynomial $\Phi(\xi, Y)$ is given by (1.1). It is easy to verify that if we substitute to the first equation the expression for Y^2 taken from the second equation, then leading powers of ξ cancel and the equation for ξ turns out to be cubic. By $\xi_i(s_1, s_2), \quad i = 1, 2, 3$ denote the roots of this equation.

Theorem. *The function*

$$\tilde{S}(s_1, s_2, e_1, e_2) = \frac{1}{4} \sum_{n=1}^{3} \left[\delta \operatorname{arctanh} \frac{\xi_n - \frac{1}{2}(s_1 + s_2)}{\delta\, Y(\xi_n)} - \int^{\xi_n} \frac{d\xi}{Y(\xi)} \right] \tag{3.1}$$

satisfies the Hamilton-Jacobi equation

$$H\left(\frac{\partial \tilde{S}}{\partial s_1}, \frac{\partial \tilde{S}}{\partial s_2}, s_1, s_2\right) = e_1, \qquad K\left(\frac{\partial \tilde{S}}{\partial s_1}, \frac{\partial \tilde{S}}{\partial s_2}, s_1, s_2\right) = e_2,$$

where the functions $H(p_1, p_2, s_1, s_2)$ and $K(p_1, p_2, s_1, s_2)$ *are defined by formulas* (1.3)-(1.5).

Proof. It follows from (2.9) that the partial derivatives of the action $\tilde{S}$ are given by:

$$\frac{\partial \tilde{S}}{\partial s_1} = \frac{J\, u}{8S(s_1)}, \qquad \frac{\partial \tilde{S}}{\partial s_2} = -\frac{J\, v}{8S(s_2)}. \tag{3.2}$$

The compatibility condition for this system has the form

$$S(s_2)\frac{\partial}{\partial s_2}(Ju) + S(s_1)\frac{\partial}{\partial s_1}(Jv) = 0.$$

The latter relation follows from the identities

$$\begin{gathered} \left(L'(y) + M_x(x, y)\right)(x - y) + 6L(y) - 2M(x, y) = 0, \\ (x - y)\left(M_y(x, y) + M_x(y, x)\right) + 4(M(x, y) - M(y, x)) = 0, \end{gathered} \tag{3.3}$$

which can be verified straightforwardly.

Consider an auxiliary Hamiltonian of the form (compare with formula (2.10))

$$\tilde{H} = p_1^3 L(s_1) + p_1^2 p_2 M(s_2, s_1) + p_1 p_2^2 M(s_1, s_2) + p_2^3 L(s_2). \tag{3.4}$$

It can easily be checked that identities (3.3) are fulfilled if the condition $\{\tilde{H}, \tilde{K}\} = 0$ holds, where

$$\tilde{K} = -(s_1 - s_2)^2 p_1 p_2.$$

It is easy to verify that if $\{p_i, s_j\} = \delta_{ij}$, then the Poisson brackets between functions

$$a_0 = p_1 + p_2, \qquad a_1 = s_1 p_1 + s_2 p_2, \qquad a_2 = s_1^2 p_1 + s_2^2 p_2 \tag{3.5}$$

are given by

$$\{a_0, a_1\} = a_0, \qquad \{a_0, a_2\} = 2a_1, \qquad \{a_1, a_2\} = a_2. \tag{3.6}$$

The expression $Q = a_1^2 - a_0 a_2$ is a Casimir function for the linear $sl(2)$-brackets (3.6). It turns out that $\tilde{K}$ rewritten in variables (3.5) coincides with Q. The Hamiltonian $\tilde{H}$ can also be expressed in terms of variables (3.5) only:

$$\tilde{H} = a_0^3 L\left(\frac{a_1}{a_0}\right) - \delta^2(\alpha a_2 + e_2 a_1 + e_1 a_0) + \frac{\delta^4}{4}\frac{a_1^2}{a_0}.$$

It is seen from this formula that the variable

$$\xi = \frac{a_1}{a_0} = \frac{p_1 s_1 + p_2 s_2}{p_1 + p_2}$$

should play a key role in a description of properties of polynomials (2.10), (3.4). Suppose $\tilde{H}(p_1, p_2) = 0$; then a solution z for equation (2.10) can be expressed in terms of ξ by the formula

$$z = \sqrt{\frac{s_2 - \xi}{s_1 - \xi}}\sqrt{\frac{S(s_1)}{S(s_2)}}. \tag{3.7}$$

Equation (2.10) is cubic with respect to ξ. It can be rewritten as (1.1), where

$$Y^2 \stackrel{\text{def}}{=} \frac{1}{\delta^2}(\xi - s_1)(\xi - s_2). \tag{3.8}$$

Our task is to find the solution of (3.2) explicitly. According to formulas (2.11), it suffices for that to solve the system of equation

$$\begin{aligned} \frac{\partial\sigma}{\partial s_1} &= \frac{J}{8S(s_1)} z = \frac{\delta}{4(s_1 - s_2)}\sqrt{\frac{s_2 - \xi}{s_1 - \xi}}, \\ \frac{\partial\sigma}{\partial s_2} &= -\frac{J}{8S(s_2)}\frac{1}{z} = -\frac{\delta}{4(s_1 - s_2)}\sqrt{\frac{s_1 - \xi}{s_2 - \xi}}. \end{aligned} \tag{3.9}$$

Here $z(s_1, s_2)$ is arbitrary root of equation (2.10) and ξ is the corresponding (see (3.7)) value of $\xi(s_1, s_2)$. The action function $\tilde{S}$ required can be obtained as the sum of the three solutions for (3.9), corresponding to the three branches of the function $\xi(s_1, s_2)$.

First, we find a function σ_0 such that (3.9) is fulfilled under condition that ξ is a parameter, which does not depend on s_1, s_2. It is not hard to see that

$$\sigma_0(\xi, s_1, s_2) = \frac{1}{4}\delta \operatorname{arctanh} \frac{\xi - \frac{1}{2}(s_1 + s_2)}{\sqrt{(\xi - s_1)(\xi - s_2)}}$$

Taking into account the formula

$$\frac{\partial}{\partial\xi}\sigma_0 = \frac{1}{4}\frac{1}{\sqrt{(\xi - s_1)(\xi - s_2)}}$$

and replacing $\sqrt{(\xi - s_1)(\xi - s_2)}$ by $Y\delta$, we obtain expression (3.1) for the action function.

Differentiating the action with respect to the parameters and taking into account that

$$\frac{\partial \tilde{S}}{\partial e_1} = t + c_1, \qquad \frac{\partial \tilde{S}}{\partial e_2} = c_2,$$

we finally get

$$dt = \sum_{n=1}^{3} \omega_1(\xi_n), \qquad 0 = \sum_{n=1}^{3} \omega_2(\xi_n). \tag{3.10}$$

Here $\omega_1,\ \omega_2$ belong to a basis

$$\omega_1(\xi) = \frac{d\xi}{Z}, \qquad \omega_2(\xi) = \frac{\xi\, d\xi}{Z}, \qquad \omega_3(\xi) = \frac{(\xi^2 - \delta^2 Y^2)d\xi}{Z}, \qquad \omega_4(\xi) = \frac{Y\, d\xi}{Z} \tag{3.11}$$

of holomorphic differentials on the curve $\Phi(\xi, Y) = 0$. In (3.11) we use the notation $Z \stackrel{\text{def}}{=} \frac{\partial \Phi}{\partial Y}$.

The differential ω_4 plays a special role. In variables (ξ, η), where

$$\eta = \xi^2 - \delta^2 Y^2, \tag{3.12}$$

the curve (1.1) becomes the following cubic

$$s_6\eta^3+s_5\eta^2\xi+\frac{1}{5}s_4\eta(\eta+4\xi^2)+\frac{1}{5}s_3\xi(\eta+2\xi^2)+\frac{1}{5}s_2(\eta+4\xi^2)+s_1\xi+s_0+\frac{4}{\delta^2}(\eta-\xi^2)(\alpha\eta+e_2\xi+e_1) = 0. \tag{3.13}$$

The holomorphic differential of the elliptic curve (3.13) coincides with ω_4 up to transformation (3.12).

It follows from (3.8) that the functions $Y(\xi_i)$ are linked by the relation

$$\frac{1}{\delta^2} = \frac{Y^2(\xi_1)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} + \frac{Y^2(\xi_2)}{(\xi_2 - \xi_3)(\xi_2 - \xi_1)} + \frac{Y^2(\xi_3)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)}, \tag{3.14}$$

which allows us to determine the two functions $s_1(t), s_2(t)$ starting from $\xi_1(t), \xi_2(t), \xi_3(t)$. Thus we have three conditions (3.10), (3.14) to determine three functions $\xi_i(t)$.

Condition (3.14) can be rewritten in the variables (ξ, η) as

$$\eta_1(\xi_2 - \xi_3) + \eta_2(\xi_3 - \xi_1) + \eta_3(\xi_1 - \xi_2) = 0.$$

This means that the corresponding points (ξ_i, η_i) belongs to the intersection of elliptic curve (3.13) and the straight line $\eta = \xi(s_1 + s_2) - s_1 s_2$.

4 Schottky-Manakov spinning top.

It is well-known (see, for example [19, 11]) that the Hamiltonian

$$H = (\vec{S}_1, A\vec{S}_1) + 2(\vec{S}_1, B\vec{S}_2) + (\vec{S}_2, A\vec{S}_2),$$

where $A = \mathrm{diag}(a_1, a_2, a_3),\ B = \mathrm{diag}(b_1, b_2, b_3)$, commutes with a quadratic polynomial K with respect to the spin Poisson brackets

$$\{S_i^\alpha, S_j^\beta\} = \kappa\, \varepsilon_{\alpha\beta\gamma} S_i^\gamma \delta_{ij} \tag{4.1}$$

iff

$$b_1^2(a_2 - a_3) + b_2^2(a_3 - a_1) + b_3^2(a_1 - a_2) + (a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = 0. \tag{4.2}$$

Here $\varepsilon_{\alpha\beta\gamma}$ is the totally skew-symmetric tensor, κ is a parameter.

Since H and K may be replaced by arbitrary linear combinations of H, K and the Casimir functions

$$J_1 = (\vec{S}_1, \vec{S}_1), \qquad J_2 = (\vec{S}_2, \vec{S}_2)$$

for the brackets (4.1), the integral K can be reduced [13] to the form

$$K = 2(\vec{S}_1, \hat{C}\vec{S}_2), \qquad C = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3).$$

Without loss of generality the matrices A and B defined the Hamiltonian H can be choosen as follows

$$A = \mathrm{diag}(-\alpha_1^2, -\alpha_2^2, -\alpha_3^2), \qquad B = \mathrm{diag}(\alpha_2\alpha_3 + \lambda\alpha_1,\ \alpha_3\alpha_1 + \lambda\alpha_2,\ \alpha_1\alpha_2 + \lambda\alpha_3),$$

The arbitrary parameter λ corresponds to the shift of H by $\lambda\, K$.

Reduction to the standard brackets. Let us fix the values of the Casimir functions: $(\vec{S}_k, \vec{S}_k) = j_k^2$. Then the formulas

$$\vec{S}_k = p_k \vec{K}(q_k) + \frac{j_k}{2}\vec{K}'(q_k), \qquad \text{where} \qquad \vec{K}(q) = ((q^2 - 1),\, i(q^2 + 1),\, 2q), \tag{4.3}$$

define a transformation of the Poisson manifold with coordinates $\vec{S}_1, \vec{S}_2$ and brackets (4.1), where $\kappa = -2i$, to the manifold with coordinates p_1, p_2, q_1, q_2 and canonical Poisson brackets $\{p_\alpha, q_\beta\} = \delta_{\alpha\beta}$. As a result of this transformation we get

$$\begin{gathered} H = p_1^2 r(q_1) + \frac{j_1}{2} p_1\, r'(q_1) + \frac{j_1^2}{12} r''(q_1) + p_2^2 r(q_2) + \frac{j_2}{2} p_2\, r'(q_2) + \frac{j_2^2}{12} r''(q_2) + \\ 2\left(p_1 + \frac{j_1}{2}\frac{\partial}{\partial q_1}\right)\left(p_2 + \frac{j_2}{2}\frac{\partial}{\partial q_2}\right) Z(q_1, q_2), \end{gathered} \tag{4.4}$$

$$K = 2\left(p_1 + \frac{j_1}{2}\frac{\partial}{\partial q_1}\right)\left(p_2 + \frac{j_2}{2}\frac{\partial}{\partial q_2}\right)W(q_1, q_2), \tag{4.5}$$

where

$$r(x) \stackrel{\text{def}}{=} (\vec{K}(x),\, A\vec{K}(x)) = -\alpha_1^2(x^2-1)^2 + \alpha_2(x^2+1)^2 - 4\alpha_3 x^2,$$

$$\begin{gathered} Z(x,y) \stackrel{\text{def}}{=} (\vec{K}(x),\, B\vec{K}(y)) = \\ (\alpha_2\alpha_3 + \lambda\alpha_1)(x^2-1)(y^2-1) - (\alpha_3\alpha_1 + \lambda\alpha_2)(x^2+1)(y^2+1) + 4(\alpha_1\alpha_2 + \lambda\alpha_3)xy, \end{gathered} \tag{4.6}$$

$$W(x,y) \stackrel{\text{def}}{=} (\vec{K}(x),\, C\vec{K}(y)) = \alpha_1(x^2-1)(y^2-1) - \alpha_2(x^2+1)(y^2+1) + 4\alpha_3 xy. \tag{4.7}$$

It is possible to check that

$$Z^2(x,y) - r(x)r(y) = W(x,y)\bar{W}(x,y), \tag{4.8}$$

where $\bar{W}$ is a polynomial quadratic in each of variables.

Remark. Note that the more general Hamiltonian

$$H = \sum_{ij,\mu,\nu} S_i^\mu S_j^\nu c_{\mu\nu}^{ij}, \qquad i,j = 1,...,N, \qquad \mu,\nu = x,y,z,$$

describing an interaction of N spins, can be reduced to

$$H = \sum_{ij} g^{ij} p_i p_j + \sum_i a^i p_i + v,$$

where

$$g^{ij} = \sum_{\mu,\nu} K^\mu(x^i) K^\nu(x^j) c_{\mu\nu}^{ij},$$

$$a^i = \sum_k j_k \frac{\partial g^{ik}}{\partial x^k} - \frac{1}{2} j_i \frac{\partial g^{ii}}{\partial x^i},$$

$$v = \frac{1}{4}\sum_{ik} j_i j_k \frac{\partial^2 g^{ik}}{\partial x^i \partial x^k} - \frac{1}{6}\sum_k j_k^2 \frac{\partial^2 g^{kk}}{\partial x^k \partial x^k}$$

by a similar transformation

$$\vec{S}_k = p_k \vec{K}(x^k) + \frac{j_k}{2}\vec{K}'(x^k), \qquad k = 1,...,N.$$

In terms of the coordinates and the velocities the Lagrangian and the energy read as

$$L = \frac{1}{4}\sum_{ij} g_{ij}(\dot{x}^i - a^i)(\dot{x}^j - a^j) - v, \qquad E = \frac{1}{4}\sum_{ij} g_{ij}(\dot{x}^i\dot{x}^j - a^i a^j) + v,$$

where $\sum_j g_{ij} g^{jk} = \delta_i^k$.

An additional integral of the form

$$K = \sum_{ij} G^{ij} p_i p_j + \sum_i A^i p_i + V$$

exists iff

$$\sum_k \frac{\partial G^{ij}}{\partial x^k} g^{kl} - \frac{\partial g^{ij}}{\partial x^k} G^{kl} = 0, \qquad \sum_k a^k \frac{\partial G^{ij}}{\partial x^k} - 2\frac{\partial a^i}{\partial x^k} G^{kj} - A^k \frac{\partial g^{ij}}{\partial x^k} + 2\frac{\partial A^i}{\partial x^k} g^{kj} = 0,$$

$$\sum_k a^k \frac{\partial A^i}{\partial x^k} - A^k \frac{\partial a^i}{\partial x^k} + 2g^{ik} \frac{\partial V}{\partial x^k} - 2G^{ik} \frac{\partial v}{\partial x^k} = 0, \qquad \sum_k a^k \frac{\partial V}{\partial x^k} - A^k \frac{\partial v}{\partial x^k} = 0.$$

Diagonalization of the quadratic part. Under transformation (4.3) the functions H and K take the form

$$H = ap_1^2 + 2bp_1p_2 + cp_2^2 + dp_1 + ep_2 + f, \tag{4.9}$$

$$K = Ap_1^2 + 2Bp_1p_2 + Cp_2^2 + Dp_1 + Ep_2 + F, \tag{4.10}$$

where the coefficients are some (in our case rational) functions of the variables q_1, q_2.

General formulas related to a pair of commuting Hamiltonians quadratic in momenta and some examples can be found in [16, 14, 15, 17].

The class of Hamiltonians (4.9) is invariant with respect to *canoniacal* transformations of the form

$$p_1 = k_1\hat{p}_1 + k_2\hat{p}_2 + k_3, \qquad p_2 = \bar{k}_1\hat{p}_1 + \bar{k}_2\hat{p}_2 + \bar{k}_3, \qquad q_1 = \phi, \qquad q_2 = \bar{\phi},$$

where $k_i, \bar{k}_i, \phi, \bar{\phi}$ are some functions of $\hat{q}_1, \hat{q}_2$. It is easily seen that the functions $k_1, k_2, \bar{k}_1, \bar{k}_2$ can be uniquely expressed through $\phi, \bar{\phi}$ from the condition $\{p_\alpha, q_\beta\} = \delta_{\alpha\beta}$:

$$k_1 = \frac{\partial \bar{\phi}}{\partial \hat{q}_2} W^{-1}, \quad k_2 = -\frac{\partial \bar{\phi}}{\partial \hat{q}_1} W^{-1}, \qquad \bar{k}_1 = -\frac{\partial \phi}{\partial \hat{q}_2} W^{-1}, \quad \bar{k}_2 = \frac{\partial \phi}{\partial \hat{q}_1},$$

where

$$W = \frac{\partial \bar{\phi}}{\partial \hat{q}_2}\frac{\partial \phi}{\partial \hat{q}_1} - \frac{\partial \bar{\phi}}{\partial \hat{q}_1}\frac{\partial \phi}{\partial \hat{q}_2}.$$

The fact that the coefficient of $\hat{p}_1\hat{p}_2$ in the transformed Hamiltonian (4.9) vanishes is equivalent to

$$a(\phi, \bar{\phi})\frac{\partial \bar{\phi}}{\partial \hat{q}_1}\frac{\partial \bar{\phi}}{\partial \hat{q}_2} + c(\phi, \bar{\phi})\frac{\partial \phi}{\partial \hat{q}_1}\frac{\partial \phi}{\partial \hat{q}_2} = b(\phi, \bar{\phi})\left(\frac{\partial \bar{\phi}}{\partial \hat{q}_2}\frac{\partial \phi}{\partial \hat{q}_1} + \frac{\partial \bar{\phi}}{\partial \hat{q}_1}\frac{\partial \phi}{\partial \hat{q}_2}\right). \tag{4.11}$$

Analogously, the condition

$$A(\phi, \bar{\phi})\frac{\partial \bar{\phi}}{\partial \hat{q}_1}\frac{\partial \bar{\phi}}{\partial \hat{q}_2} + C(\phi, \bar{\phi})\frac{\partial \phi}{\partial \hat{q}_1}\frac{\partial \phi}{\partial \hat{q}_2} = B(\phi, \bar{\phi})\left(\frac{\partial \bar{\phi}}{\partial \hat{q}_2}\frac{\partial \phi}{\partial \hat{q}_1} + \frac{\partial \bar{\phi}}{\partial \hat{q}_1}\frac{\partial \phi}{\partial \hat{q}_2}\right) \tag{4.12}$$

guarantees that the coefficient of $\hat{p}_1\hat{p}_2$ in the transformed integral (4.10) is equal to zero. The existence of a solution $\phi, \bar{\phi}$ for system (4.11), (4.12) is obvious. Our goal is to find the functions $\phi, \bar{\phi}$ for the pair (4.4), (4.5) explicitly.

Let us introduce the Kowalewski variables $s_1(q_1, q_2)$ and $s_2(q_1, q_2)$ as the roots of the equation

$$W(q_1, q_2)\, s^2 + 2Z(q_1, q_2)\, s + \bar{W}(q_1, q_2) = 0,$$

where W and Z are given by (4.7),(4.6) and the polynomial $\bar{W}$ is defined by (4.8).

Proposition 3. *In the variables s_1, s_2 the functions H and K have the form* (1.3)-(1.5), *where polynomials S and R are defined by the formulas*

$$S(x) = -(x+\lambda-\alpha_1-\alpha_2-\alpha_3)(x+\lambda+\alpha_1+\alpha_2-\alpha_3)(x+\lambda+\alpha_1-\alpha_2+\alpha_3)(x+\lambda-\alpha_1+\alpha_2+\alpha_3)$$

and

$$\alpha = \frac{1}{4}\left(\frac{\delta^2}{5} - \nu^2\right), \qquad \delta = j_1 - j_2, \qquad \nu = j_1 + j_2.$$

The values h and k of the integrals H and K are related to the constants e_1, e_2 from (1.2) by

$$h = e_1 + \frac{1}{12}\alpha S''(0), \qquad k = e_2 + \frac{1}{12}\alpha S'''(0).$$

Thus the general scheme from Section 3 can be applied to the Schottky-Manakov spinning top.

5 The Kowalewski gyrostat.

In this Section we use some formulas from [21] that describe a dynamical system for the Kowalewski gyrostat in the Kowalewski variables.

The Hamiltonain structure for the gyrostat is defined by the $e(3)$-Poisson brackets

$$\{M_i\,, M_j\,\} = \varepsilon_{ijk} M_k\,, \qquad \{M_i\,, \gamma_j\,\} = \varepsilon_{ijk}\gamma_k\,, \qquad \{\gamma_i\,, \gamma_j\,\} = 0,$$

where ε_{ijk} is the totally skew-symmetric tensor. The brackets possess the Casimir functions

$$A = \sum_{k=1}^{3} \gamma_k^2, \qquad B = \sum_{k=1}^{3} \gamma_k M_k. \tag{5.1}$$

The Hamiltonian for the gyrostat is as follows:

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2 - 2\lambda M_3) + c\gamma_1\,, \tag{5.2}$$

where c and λ are constants. An additional integral of motion is given by

$$K = \xi_1\xi_2 + 4\lambda\Big((M_3-\lambda)z_1z_2 - (z_1+z_2)c\gamma_3\Big), \tag{5.3}$$

where

$$\xi_1 = z_1^2 - 2c(\gamma_1 + i\gamma_2)\,, \qquad \xi_2 = z_2^2 - 2c(\gamma_1 - i\gamma_2)$$

and

$$z_1 = M_1 + iM_2, \qquad z_2 = M_1 - iM_2.$$

We put

$$R(z_1, z_2) = z_1^2 z_2^2 - 2h(z_1^2 + z_2^2) - 4cb(z_1 + z_2) - 4c^2\,a + k.$$

Here a, b, h and k are values of integrals (5.1), (5.2), and (5.3). The Kowalewski variables are defined as follows

$$s_{1,2} = \frac{R(z_1,z_2) \pm \sqrt{R(z_1,z_1)R(z_2,z_2)}}{2(z_1-z_2)^2}.$$

It turns out [21] that

$$h = \frac{s_1 - s_2}{2}\left(\frac{\dot{s}_1^2}{\varphi_1} - \frac{\dot{s}_2^2}{\varphi_2}\right) - \frac{s_1+s_2}{2}\,, \tag{5.4}$$

$$\frac{k}{4} = (2h + s_1 + s_2)\lambda^2 - \lambda\sqrt{-\varphi_1\varphi_2}\left(\frac{\dot{s}_1}{\varphi_1} + \frac{\dot{s}_2}{\varphi_2}\right) + (s_1 - s_2)\left(\frac{s_2\dot{s}_1^2}{\varphi_1} - \frac{s_1\dot{s}_2^2}{\varphi_2}\right) - s_1s_2 + h^2\,. \tag{5.5}$$

Here $\varphi_i = S(s_i)$,

$$S(s) = 4s^3 - 8h\,s^2 + 4h^2\,s - k\,s + 4c^2a\,s + 4c^2\,b.$$

Substituting the expressions (2.8)

$$\dot{s}_1 = -\frac{1}{4}\frac{J}{s_1 - s_2}(u+1), \qquad \dot{s}_2 = -\frac{1}{4}\frac{J}{s_1 - s_2}(v+1)$$

in (5.4), (5.5) for the velocities, we get just (2.5), where

$$k(x) = 4(x+h)^2 + 4\delta^2(x - 2h) - k, \qquad \lambda = i\delta.$$

Thus in this case the scheme for the separation of variables described in Section 3 is applicable. The curve $\Phi(Y,\xi) = 0$ is of genus 3. The differentials $\omega_1, \omega_2, \omega_4$ form a basis of holomorphic differentials.

Notice that the substitution $\delta = -i\lambda$ in (3.1) results the change *arctanh* $\to$ *arctan* and the action remains to be real.

6 The Clebsch spinning top.

The Clebsch spinning top is defined by the Hamiltonian

$$H = \frac{1}{2}(J_1^2 + J_2^2 + J_3^2) + \frac{1}{2}(\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2)$$

which commutes with respect to the $e(3)$-Poisson brackets

$$\{J_i, J_j\} = i\,\varepsilon_{ijk} J_k, \qquad \{x_i, x_j\} = 0, \qquad \{J_i, x_j\} = i\,\varepsilon_{ijk} x_k$$

with the first integral

$$K = (\lambda_1 J_1^2 + \lambda_2 J_2^2 + \lambda_3 J_3^2) - \lambda_1\lambda_2\lambda_3(\frac{x_1^2}{\lambda_1} + \frac{x_2^2}{\lambda_2} + \frac{x_3^2}{\lambda_3}).$$

Let us fix the values of the Casimir functions as follows

$$x_1^2 + x_2^2 + x_3^2 = a^2, \qquad J_1 x_1 + J_2 x_2 + J_3 x_3 = l.$$

Using the parameterization

$$J_1 = \frac{1}{2}p_1(1-q_1^2)+\frac{1}{2}p_2(1-q_2^2)+\frac{l}{a}\,q_1,\ J_2 = \frac{i}{2}p_1(1+q_1^2)+\frac{i}{2}p_2(1+q_2^2)-i\frac{l}{a}\,q_1,\ J_3 = p_1q_1+p_2q_2-\frac{l}{a},$$

$$x_1 = a\frac{1 - q_1 q_2}{q_1 - q_2}, \qquad x_2 = ia\frac{1 + q_1 q_2}{q_1 - q_2}, \qquad x_3 = a\frac{q_1 + q_2}{q_1 - q_2},$$

we express H and K in terms of canonically conjugated variables $p_1, q_1,\ p_2, q_2$:

$$H = -\frac{1}{2}(x-y)^2 p_1 p_2 + \frac{l}{a}\,p_2(x-y) + \frac{a^2}{2}\frac{W(x,y)}{(x-y)^2} + \frac{1}{2}(\lambda a^2 + \frac{l^2}{a^2})$$

$$K = \frac{1}{4}(R(x)p_1^2+R(y)p_2^2+2\,p_1p_2W(x,y))-\frac{l}{4a}p_1R'(x)-\frac{l}{2a}p_2W_x(x,y)-a^2\frac{\bar{W}(x,y)}{(x-y)^2}+\frac{l^2R''(x)}{12a^2}+\frac{\lambda l^2}{3a^2}$$

where

$$R(x) = (\lambda_1 - \lambda_2)(x^2-1)^2 + 4(\lambda_3 - \lambda_2)x^2,$$

$$W(x,y) = \lambda_1(x^2-1)(y^2-1) - \lambda_2(x^2+1)(y^2+1) + 4\lambda_3 xy,$$

$$\bar{W}(x,y) = \lambda_2\lambda_3(x^2-1)(y^2-1) - \lambda_3\lambda_1(x^2+1)(y^2+1) + 4\lambda_1\lambda_2 xy + \kappa(x-y)^2,$$

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3, \qquad \kappa = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1.$$

We define Kowalewski variables as the roots $s_1(q_1,q_2)$ and $s_2(q_1,q_2)$ of the equation

$$(q_1 - q_2)^2\, s^2 + W(q_1,q_2)\, s + \bar{W}(q_1,q_2) = 0.$$

Associating the t-dynamics with K, we get equations (2.3), (2.4) for s_1, s_2, where

$$S(\xi) = 4(\xi - \lambda_1)(\xi - \lambda_2)(\xi - \lambda_3).$$

Our general procedure for separation of variables from Section 3 leads to curve (1.1), where

$$s_6 = 0, \qquad l(\xi) = \frac{l^2}{64}, \qquad 4\,k(\xi) = a^2\xi^2 + (2e_1 - a^2\lambda)\xi - e_2.$$

Note that for the Clebsch case we have $\delta = i\frac{l}{4a}$. One has to replace *arctanh* by *arctan* in the formula (3.1) and after that the action remains to be real.

7 Case $S = -1$

In the case $S(\xi) = -1$ common solutions to the pair (1.3), (1.5) satisfy the Gibbons-Tsarev equation (2.7). In this case the curve (1.1) has the form

$$-4\alpha\delta^2 Y^4 + (4e_1 + 4\xi e_2 + 4\xi^2\alpha)Y^2 + 1 = 0.$$

This curve of genus 1 can be parameterized by the Weierstrass function as follows:

$$Y = -\frac{1}{16\alpha^2}\frac{y}{x - \beta_3}, \qquad \xi = -\frac{e_2}{2\alpha} - \frac{y}{16}\frac{\delta}{\alpha^2}\left(\frac{1}{x-\beta_1} + \frac{1}{x-\beta_2} - \frac{1}{x-\beta_3}\right),$$

where $x = \wp(z, g_2, g_3), \quad y = \wp'(z, g_2, g_3)$,

$$y^2 = 4(x-\beta_1)(x-\beta_2)(x-\beta_3), \quad \beta_3 = \frac{8\alpha^2}{3\delta^2}(4\alpha e_1 - e_2^2),\ \beta_{1,2} = -\frac{1}{2}\beta_3 \pm \frac{16\alpha^3}{\delta}\sqrt{-\alpha}.$$

The meromorphic integrals in

$$dt = \sum_{n=1}^{3}\omega_1(\xi_n), \qquad d\tau = \sum_{n=1}^{3}\omega_2(\xi_n)$$

can be taken. The result is given by

$$z_1 + z_2 + z_3 = const, \quad t = \frac{1}{2\sqrt{-\alpha}}\sum_{i=1}^{3}\log\left(\frac{x_i - \beta_2}{\beta_1 - x_i}\right), \quad \tau = -\frac{e_2}{\alpha}t - 2\alpha\sum_{i=1}^{3}\frac{y_i}{(x_i-\beta_1)(x_i-\beta_2)}$$

where $x_i = \wp(z_i, g_2, g_3),\ y_i = \wp'(z_i, g_2, g_3)$.

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